

On the uniform convergence of double sine series

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Abstract

The fundamental theorem in the theory of the uniform convergence of sine series is due to Chaundy and Jolliffe from 1916 (see [1]). Several authors gave conditions for this problem supposing that coefficients are monotone, non-negative or more recently, general monotone (see [8], [6] and [2], for example). There are also results for the regular convergence of double sine series to be uniform in case the coefficients are monotone or general monotone double sequences. In this article we give new sufficient conditions for the uniformity of the regular convergence of double sine series, which are necessary as well in case the coefficients are non-negative. We shall generalize those results defining a new class of double sequences for the coefficients.

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1 Known results: uniform convergence of single sine series

Let $\{c_k\}_{k=1}^{\infty}$ be a non-negative real sequence and consider the series

$$\sum_{k=1}^{\infty} c_k \sin kx. \quad (1.1)$$

In 1916 Chaundy and Jolliffe [1] proved the following classical result.

Theorem 1. *If $\{c_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ is decreasing to zero, then (1.1) converges uniformly in x if and only if*

$$kc_k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \quad (1.2)$$

Several classes of sequences have been introduced to generalize Theorem 1 (see [10], [9], [2], [6]). These classes are larger than the class of monotone sequences and contain sequences of complex numbers as well. The definitions of the latest classes are the following:

$$\begin{aligned} MVBVS &= \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists \mathcal{C} > 0, \lambda \geq 2 : \sum_{k=n}^{2n} |\Delta_1 c_k| \leq \frac{\mathcal{C}}{n} \sum_{k=\lceil n/\lambda \rceil}^{\lceil \lambda n \rceil} |c_k|, n \geq \lambda \right\}, \\ SBVS &= \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists \mathcal{C} > 0, \lambda \geq 2 : \sum_{k=n}^{2n-1} |\Delta_1 c_k| \leq \frac{\mathcal{C}}{n} \left(\sup_{m \geq \lceil n/\lambda \rceil} \sum_{k=m}^{2m} |c_k| \right), n \geq \lambda \right\}, \\ SBVS_2 &= \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists \mathcal{C} > 0, b(k) \nearrow \infty : \sum_{k=n}^{2n-1} |\Delta_1 c_k| \leq \frac{\mathcal{C}}{n} \left(\sup_{m \geq b(n)} \sum_{k=m}^{2m} |c_k| \right), n \geq 1 \right\}, \\ GM(\beta, r) &= \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists \mathcal{C} > 0 : \sum_{k=n}^{2n-1} |\Delta_r a_k| \leq \mathcal{C} \beta_n, n \geq 1 \right\}, \end{aligned}$$

where $\Delta_r c_k = c_k - c_{k+r}$ for $r \in \mathbb{N}$, and the constants \mathcal{C} and λ depend only on $\{c_k\}$, a sequence $\{b(k)\}_{k=1}^{\infty} \subset \mathbb{R}_+$ is increasing and $\beta := (\beta_k)$ is a non-negative sequence. It was proved in [2] that $MVBVS \subsetneq SBVS \subsetneq SBVS_2$ and a series (1.1) with coefficients of complex numbers from the classes $MVBVS$, $SBVS$, $SBVS_2$, is uniformly convergent if (1.2) is satisfied. In [6] Szal showed that the class $GM(\beta^*, r)$, with $\beta^* = \beta_n = \frac{1}{n} \sum_{k=\lceil n/\lambda \rceil}^{\lceil \lambda n \rceil} |c_k|$ ($\lambda > 1$) and $r = 2$ is larger than $MVBVS$ and a series (1.1) with coefficients from $GM(\beta^*, 2)$ is uniformly convergent if (1.2) holds. However, the necessity of condition (1.2) for the uniform convergence of (1.1) is proved for sine series with coefficients of non-negative numbers from one of the above classes.

2 Known results: uniform convergence of double sine series

We start this section by giving some definitions and notations.

A double series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk}$$

of complex numbers converge regularly if the sum

$$\sum_{j=1}^m \sum_{k=1}^n c_{jk}$$

converge to a finite number as m and n tend to infinity independently of each other, moreover, both the column series and row series

$$\sum_{j=1}^{\infty} z_{jn}, \quad n = 1, 2, \dots, \quad \text{and} \quad \sum_{k=1}^{\infty} z_{mk} \quad m = 1, 2, \dots$$

are convergent. Or equivalently, if for any $\epsilon > 0$ there exists a positive number $m_0 = m_0(\epsilon)$ such that

$$\left| \sum_{j=m}^M \sum_{k=n}^N z_{jk} \right| < \epsilon$$

holds for any m, n, M, N for which $m + n > m_0$, $1 \leq m \leq M$ and $1 \leq n \leq N$.

A monotonically decreasing double sequence $\{c_{jk}\}_{j,k=1}^{\infty}$ is a sequence of real numbers such that

$$\Delta_{10} c_{jk} \geq 0, \quad \Delta_{01} c_{jk} \geq 0, \quad \Delta_{11} c_{jk} \geq 0, \quad j, k = 1, 2, \dots,$$

where

$$\begin{aligned} \Delta_{10} c_{jk} &:= c_{jk} - c_{j+1,k}, & \Delta_{01} c_{jk} &:= c_{jk} - c_{j,k+1}, \\ \Delta_{11} c_{jk} &:= \Delta_{10}(\Delta_{01} c_{jk}) = \Delta_{01}(\Delta_{10} c_{jk}) = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}. \end{aligned}$$

Let $\{c_{jk}\}_{j,k=1}^{\infty}$ be a double sequence of complex numbers. Consider the double sine series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk} \sin jx \sin ky. \tag{2.1}$$

The two-dimensional extension of Theorem 1 is due to Žak and Šneider.

Theorem 2. ([11]) *If $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{R}_+$ is a monotonicity decreasing double sequence, then (2.1) is uniformly regularly convergent in (x, y) if and only if*

$$jk c_{jk} \rightarrow 0 \quad \text{as} \quad j + k \rightarrow \infty. \tag{2.2}$$

Theorem 2 was generalized by Kórus and Móricz [3] and by Kórus [4] (and also by Leindler [5]). They have defined new classes of double sequences to obtain those generalizations. We present below those definitions and their results.

Definition 1. ([3]) A double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to the class $MVBVDS$, if there exist positive constants \mathcal{C} and $\lambda \geq 2$, depending only on $\{c_{jk}\}$, such that:

$$\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| \leq \frac{\mathcal{C}}{m} \sum_{[j=m/\lambda]}^{[\lambda m]} |c_{jn}|, \quad m \geq \lambda, \quad n \geq 1, \quad (2.3)$$

$$\sum_{k=n}^{2n-1} |\Delta_{01} c_{km}| \leq \frac{\mathcal{C}}{n} \sum_{[k=n/\lambda]}^{[\lambda n]} |c_{km}|, \quad n \geq \lambda, \quad m \geq 1, \quad (2.4)$$

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| \leq \frac{\mathcal{C}}{mn} \sum_{[j=m/\lambda]}^{[\lambda m]} \sum_{[k=n/\lambda]}^{[\lambda n]} |c_{jk}|, \quad m, n \geq \lambda. \quad (2.5)$$

Theorem 3. ([3])

- (i) If $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to the class $MVBVDS$ and (2.2) holds, then (2.1) converges regularly, uniformly in (x, y) .
- (ii) Conversely, if $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{R}_+$ belongs to the class $MVBVDS$ and (2.1) is uniformly regularly convergent in (x, y) , then (2.2) is satisfied.

Definition 2. ([4]) A double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to the class $SBVDS_1$, if there exist positive constants \mathcal{C} and integer $\lambda \geq 2$ and sequences $\{b_1(l)\}_{l=1}^{\infty}$, $\{b_2(l)\}_{l=1}^{\infty}$, $\{b_3(l)\}_{l=1}^{\infty}$, each one tends (not necessarily monotonically) to infinity, all of them depending only on $\{c_{jk}\}$, such that:

$$\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| \leq \frac{\mathcal{C}}{m} \left(\max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} |c_{jn}| \right), \quad m \geq \lambda, \quad n \geq 1, \quad (2.6)$$

$$\sum_{k=n}^{2n-1} |\Delta_{01} c_{km}| \leq \frac{\mathcal{C}}{n} \left(\max_{b_2(n) \leq M \leq \lambda b_2(n)} \sum_{k=N}^{2N} |c_{km}| \right), \quad n \geq \lambda, \quad m \geq 1, \quad (2.7)$$

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| \leq \frac{\mathcal{C}}{mn} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right), \quad m, n \geq \lambda. \quad (2.8)$$

Theorem 4. ([4]) If $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{R}_+$ belongs to the class $SBVDS_1$ and (2.1) is uniform regularly convergent in (x, y) , then (2.2) is satisfied.

Definition 3. ([4]) A double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to the class $SBVDS_2$, if there exist positive constants \mathcal{C} and integer $\lambda \geq 1$ and sequence $\{b(l)\}_{l=1}^{\infty}$ tending monotonically to infinity, depending only on $\{c_{jk}\}$, for which:

$$\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| \leq \frac{\mathcal{C}}{m} \left(\sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}| \right), \quad m \geq \lambda, \quad n \geq 1, \quad (2.9)$$

$$\sum_{k=n}^{2n-1} |\Delta_{01} c_{km}| \leq \frac{\mathcal{C}}{n} \left(\sup_{N \geq b(n)} \sum_{k=N}^{2N} |c_{km}| \right), \quad n \geq \lambda, \quad m \geq 1, \quad (2.10)$$

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11} c_{jk}| \leq \frac{\mathcal{C}}{mn} \left(\sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right), \quad m, n \geq \lambda. \quad (2.11)$$

Theorem 5. ([4]) If $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to the class $SBVDS_2$ and (2.2) holds, then the regularly convergent of (2.1) is uniform in (x, y) .

Theorem 6. ([4]) $MVBVDS \subsetneq SBVDS_1 \subsetneq SBVDS_2$.

Now, we shall define a new class of double sequences in the following way: Let, for $r \in \mathbb{N}$,

$$\Delta_{r0} c_{jk} := c_{jk} - c_{j+r,k}, \quad \Delta_{0r} c_{jk} := c_{jk} - c_{j,k+r}$$

and

$$\Delta_{rr} c_{jk} := \Delta_{r0}(\Delta_{0r} c_{jk}).$$

Definition 4. A double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to the class $DGM(\alpha, \beta, \gamma, r)$ (called Double General Monotone), if there exist positive constants \mathcal{C} and integer λ depending only on $\{c_{jk}\}$, for which:

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{r0} c_{jn}| &\leq \mathcal{C} \alpha_{mn}, \quad m \geq \lambda, \quad n \geq 1, \\ \sum_{k=n}^{2n-1} |\Delta_{0r} c_{km}| &\leq \mathcal{C} \beta_{mn}, \quad n \geq \lambda, \quad m \geq 1, \\ \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{rr} c_{jk}| &\leq \mathcal{C} \gamma_{mn}, \quad m, n \geq \lambda. \end{aligned}$$

hold, where $\alpha := \{\alpha_{mn}\}_{m,n=1}^{\infty}$, $\beta := \{\beta_{mn}\}_{m,n=1}^{\infty}$, $\gamma := \{\gamma_{mn}\}_{m,n=1}^{\infty}$ are non-negative double sequences and $r \in \mathbb{N}$.

Using our definition for $r = 1$, we have:

- 1) $MVBVDS \equiv DGM({}_1\alpha, {}_1\beta, {}_1\gamma, 1)$, where $\{{}_1\alpha\}$, $\{{}_1\beta\}$ and $\{{}_1\gamma\}$ are the sequences defined by the formulas on the right sides of the inequalities (2.3), (2.4) and (2.5), respectively;
- 2) $SBVDS_1 \equiv DGM({}_2\alpha, {}_2\beta, {}_2\gamma, 1)$, where $\{{}_2\alpha\}$, $\{{}_2\beta\}$ and $\{{}_2\gamma\}$ are the sequences defined by the formulas on the right sides of the inequalities (2.6), (2.7) and (2.8), respectively;
- 3) $SBVDS_2 \equiv DGM({}_3\alpha, {}_3\beta, {}_3\gamma, 1)$, where $\{{}_3\alpha\}$, $\{{}_3\beta\}$ and $\{{}_3\gamma\}$ are the sequences defined by the formulas on the right sides of the inequalities (2.9), (2.10) and (2.11); respectively.

In this paper we shall present some properties of the classes $DGM({}_2\alpha, {}_2\beta, {}_2\gamma, 2)$ and $DGM({}_3\alpha, {}_3\beta, {}_3\gamma, 2)$. Moreover, we generalize and extend the results of Kórus ([4]) to the classes $DGM({}_2\alpha, {}_2\beta, {}_2\gamma, 2)$ or $DGM({}_3\alpha, {}_3\beta, {}_3\gamma, 2)$, respectively.

3 Auxiliary results

Lemma 1. If $\{c_{jk}\} \subset \mathbb{C}$ is such that condition (2.2) and the inequality

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22} c_{jk}| \leq \frac{\mathcal{C}}{mn} \left(\sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right), \quad m, n \geq \lambda \quad (3.1)$$

are satisfied, then

$$mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22} c_{jk}| \rightarrow 0 \quad \text{as} \quad m+n \rightarrow \infty \quad m, n \geq \lambda.$$

Proof. Set $\epsilon > 0$ arbitrarily. By condition (2.2) and from the fact that $\{b(l)\}$ tends monotonically to infinity, there exists an $m_1 = m_1(\epsilon)$ such that

$$jk|c_{jk}| < \epsilon \quad \text{for all } j, k \quad j+k > b(m_1).$$

Then, by (3.1), assuming $m+n > m_1$ and $m, n \geq \lambda$, we have

$$\begin{aligned} \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22} c_{jk}| &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{j=2^r m}^{2^{r+1}m-1} \sum_{k=2^s n}^{2^{s+1}n-1} |\Delta_{22} c_{jk}| \leq \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\mathcal{C}}{2^r m 2^s n} \left(\sup_{M+N \geq b(2^r m + 2^s n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right) \\ &< \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\mathcal{C}\epsilon}{2^r m 2^s n} \left(\sup_{M+N \geq b(2^r m + 2^s n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \frac{1}{jk} \right) \leq \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{4\mathcal{C}\epsilon}{2^r m 2^s n} \leq \frac{16\mathcal{C}\epsilon}{mn}, \end{aligned} \quad (3.2)$$

since

$$\sum_{j=M}^{2M} \sum_{k=N}^{2N} \frac{1}{jk} \leq \sum_{j=M}^{2M} \sum_{k=N}^{2N} \frac{1}{MN} \leq \frac{2M \cdot 2N}{MN} = 4 \quad \text{for any } M, N.$$

This complete the proof. \square

Lemma 2. Let $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ and the condition (2.2) is satisfied.

(i) If the inequality

$$\sum_{j=m}^{2m-1} |\Delta_{20} c_{jn}| \leq \frac{\mathcal{C}}{m} \left(\sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}| \right), \quad m \geq \lambda, \quad n \geq 1 \quad (3.3)$$

holds, then

$$m \sup_{k \geq n} k \sum_{j=m}^{\infty} |\Delta_{20} c_{jk}| \rightarrow 0, \quad (3.4)$$

(ii) and if the inequality

$$\sum_{k=n}^{2n-1} |\Delta_{02} c_{km}| \leq \frac{\mathcal{C}}{n} \left(\sup_{N \geq b(n)} \sum_{k=N}^{2N} |c_{km}| \right), \quad n \geq \lambda, \quad m \geq 1 \quad (3.5)$$

holds, then

$$n \sup_{j \geq m} j \sum_{k=n}^{\infty} |\Delta_{02} c_{jk}| \rightarrow 0 \quad (3.6)$$

as $m + n \rightarrow \infty$, where $m \geq \lambda$ and $n \geq 1$ or $n \geq \lambda$ and $m \geq 1$, respectively.

Proof. Set $\epsilon > 0$ arbitrarily. By condition (2.2) and from the fact that $\{b(l)\}$ tends monotonically to infinity, there exists an $m_1 = m_1(\epsilon)$ such that

$$jk|c_{jk}| < \epsilon \quad \text{for all } j, k \quad j + k > b(m_1).$$

Part (i) By (3.3), assuming $m > \max\{m_1, \lambda\}$ and $n > m_1$, we have

$$\begin{aligned} \sup_{k \geq n} k \sum_{j=m}^{\infty} |\Delta_{20} c_{jk}| &= \sup_{k \geq n} k \sum_{r=0}^{\infty} \sum_{j=2^r m}^{2^{r+1} m - 1} |\Delta_{20} c_{jk}| \leq \sup_{k \geq n} k \sum_{r=0}^{\infty} \frac{\mathcal{C}}{2^r m} \left(\sup_{M \geq b(2^r m)} \sum_{j=M}^{2M} |c_{jk}| \right) \\ &\leq \sup_{k \geq n} k \frac{\mathcal{C}}{m} \sum_{r=0}^{\infty} \frac{1}{2^r} \left(\sup_{M \geq b(2^r m)} \sum_{j=M}^{2M} jk|c_{jk}| \frac{1}{jk} \right) < \frac{\mathcal{C}\epsilon}{m} \sum_{r=0}^{\infty} \frac{1}{2^r} \left(\sup_{M \geq b(2^r m)} \sum_{j=M}^{2M} \frac{1}{j} \right) \leq \frac{4\mathcal{C}\epsilon}{m}, \end{aligned}$$

since

$$\sum_{j=M}^{2M} \frac{1}{j} \leq \frac{M+1}{M} \leq 2.$$

This implies that (3.4) holds.

Part (ii) Using (3.5) and assuming $n > \max\{m_1, \lambda\}$ and $m > m_1$, we get

$$\begin{aligned} \sup_{m \geq j} j \sum_{k=n}^{\infty} |\Delta_{02} c_{jk}| &= \sup_{m \geq j} j \sum_{r=0}^{\infty} \sum_{k=2^r n}^{2^{r+1} n - 1} |\Delta_{02} c_{jk}| \leq \sup_{m \geq j} j \sum_{r=0}^{\infty} \frac{\mathcal{C}}{2^r n} \left(\sup_{N \geq b(2^r n)} \sum_{k=N}^{2N} |c_{jk}| \right) \\ &\leq \sup_{m \geq j} j \frac{\mathcal{C}}{n} \sum_{r=0}^{\infty} \frac{1}{2^r} \left(\sup_{N \geq b(2^r n)} \sum_{k=N}^{2N} jk|c_{jk}| \frac{1}{jk} \right) < \frac{\mathcal{C}\epsilon}{n} \sum_{r=0}^{\infty} \frac{1}{2^r} \left(\sup_{N \geq b(2^r n)} \sum_{k=N}^{2N} \frac{1}{k} \right) \leq \frac{4\mathcal{C}\epsilon}{n}. \end{aligned}$$

Hence (3.6) is satisfied.

Now, our proof is complete. \square

Lemma 3. If $\{c_{jk}\}_{j,k=1}^{\infty}$ is a non-negative sequence belonging to the class $DGM(2\alpha, 2\beta, 2\gamma, 2)$ with \mathcal{C} , λ and $\{b_1(l)\}_{l=1}^{\infty}$, $\{b_2(l)\}_{l=1}^{\infty}$, $\{b_3(l)\}_{l=1}^{\infty}$ then for any $m, n \geq \lambda$

$$mnc_{mn} \leq \mathcal{C} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) + 2\mathcal{C} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n+1} c_{jk} + 2\mathcal{C} \sum_{j=m}^{2m+1} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 8 \sum_{j=m}^{2m+1} \sum_{k=n}^{2n+1} c_{jk}.$$

Proof. Let $m, n \geq \lambda$. If $\{c_{jk}\}_{j,k=1}^\infty \in DGM(2\alpha, 2\beta, 2\gamma, 2)$, then for any v and $m \leq \mu \leq 2m$

$$\begin{aligned} c_{mv} &= \sum_{j=m}^{\mu-1} \Delta_{20} c_{jv} + c_{\mu v} + c_{\mu+1,v} - c_{m+1,v} \leq \sum_{j=m}^{2m-1} |\Delta_{20} c_{jv}| + c_{\mu v} + c_{\mu+1,v} \\ &\leq \frac{\mathcal{C}}{m} \left(\max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} c_{jv} \right) + c_{\mu v} + c_{\mu+1,v}. \end{aligned} \quad (3.7)$$

By an analogous argument, we get that for any μ and $n \leq v \leq 2n$

$$\begin{aligned} c_{\mu n} &= \sum_{k=n}^{v-1} \Delta_{02} c_{\mu k} + c_{\mu v} + c_{\mu, v+1} - c_{\mu, n+1} \leq \sum_{k=n}^{2n-1} |\Delta_{02} c_{\mu k}| + c_{\mu v} + c_{\mu, v+1} \\ &\leq \frac{\mathcal{C}}{n} \left(\max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} c_{\mu k} \right) + c_{\mu v} + c_{\mu, v+1}. \end{aligned} \quad (3.8)$$

For μ, v such that $m \leq \mu \leq 2m$ and $n \leq v \leq 2n$ we have

$$\begin{aligned} \sum_{j=m}^{\mu-1} \sum_{k=n}^{v-1} \Delta_{22} c_{jk} &= \sum_{j=m}^{\mu-1} \sum_{k=n}^{v-1} (c_{jk} - c_{j+2,k} - (c_{j,k+2} - c_{j+2,k+2})) \\ &= \sum_{k=n}^{v-1} (c_{mk} - c_{\mu k} - c_{\mu+1,k} + c_{m+1,k} - c_{m,k+2} + c_{\mu,k+2} + c_{\mu+1,k+2} - c_{m+1,k+2}) \\ &= \sum_{k=n}^{v-1} (((c_{mk} - c_{m,k+2}) + (c_{m+1,k} - c_{m+1,k+2})) - ((c_{\mu k} - c_{\mu,k+2}) + (c_{\mu+1,k} - c_{\mu+1,k+2}))) \\ &= c_{mn} - c_{mv} - c_{m,v+1} + c_{m,n+1} + c_{m+1,n} - c_{m+1,v} - c_{m+1,v+1} + c_{m+1,n+1} - c_{\mu n} \\ &\quad + c_{\mu v} + c_{\mu, v+1} - c_{\mu, n+1} - c_{\mu+1,n} + c_{\mu+1,v} + c_{\mu+1,v+1} - c_{\mu+1,n+1} \end{aligned}$$

and applying the inequality (3.1), we get

$$\begin{aligned} c_{mn} &= \sum_{j=m}^{\mu-1} \sum_{k=n}^{v-1} \Delta_{22} c_{jk} + c_{mv} + c_{m,v+1} - c_{m,n+1} - c_{m+1,n} + c_{m+1,v} + c_{m+1,v+1} - c_{m+1,n+1} \\ &\quad + c_{\mu n} - c_{\mu v} - c_{\mu, v+1} + c_{\mu, n+1} + c_{\mu+1,n} - c_{\mu+1,v} - c_{\mu+1,v+1} + c_{\mu+1,n+1} \\ &\leq \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22} c_{jk}| + c_{mv} + c_{m,v+1} + c_{m+1,v} + c_{m+1,v+1} + c_{\mu n} + c_{\mu, n+1} + c_{\mu+1,n} + c_{\mu+1,n+1} \\ &\leq \frac{\mathcal{C}}{mn} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) \\ &\quad + c_{mv} + c_{m,v+1} + c_{m+1,v} + c_{m+1,v+1} + c_{\mu n} + c_{\mu, n+1} + c_{\mu+1,n} + c_{\mu+1,n+1}. \end{aligned} \quad (3.9)$$

Adding up all inequalities in (3.9) for $\mu = m+1, m+2, \dots, 2m$ and $v = n+1, n+2, \dots, 2n$ we obtain

$$\begin{aligned} \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} c_{\mu n} &\leq \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} \frac{\mathcal{C}}{mn} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) \\ &\quad + \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} c_{mv} + \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} c_{m,v+1} + \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} c_{m+1,v} + \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} c_{m+1,v+1} \\ &\quad + \sum_{v=n+1}^{2n} \sum_{\mu=m+1}^{2m} c_{\mu n} + \sum_{v=n+1}^{2n} \sum_{\mu=m+1}^{2m} c_{\mu, n+1} + \sum_{v=n+1}^{2n} \sum_{\mu=m+1}^{2m} c_{\mu+1,n} + \sum_{v=n+1}^{2n} \sum_{\mu=m+1}^{2m} c_{\mu+1,n+1} \end{aligned}$$

$$\begin{aligned}
 &= \mathcal{C} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) + \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} (c_{\mu v} + c_{\mu+1,v}) + \sum_{\mu=m+1}^{2m} \sum_{v=n+1}^{2n} (c_{\mu,v+1} + c_{\mu+1,v+1}) \\
 &+ \sum_{v=n+1}^{2n} \sum_{\mu=m+1}^{2m} (c_{\mu n} + c_{\mu,n+1}) + \sum_{v=n+1}^{2n} \sum_{\mu=m+1}^{2m} (c_{\mu+1,n} + c_{\mu+1,n+1}).
 \end{aligned}$$

Analogously as in (3.7) and (3.8) we can obtain the following inequalities

$$c_{\mu v} + c_{\mu+1,v} \leq \frac{\mathcal{C}}{m} \left(\max_{b_1(\mu) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} c_{jv} \right) + c_{\mu v} + c_{\mu+1,v} \quad \text{for any } v \in \mathbb{N} \text{ and } \mu = m, \dots, 2m,$$

$$c_{\mu n} + c_{\mu,n+1} \leq \frac{\mathcal{C}}{n} \left(\max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} c_{\mu k} \right) + c_{\mu n} + c_{\mu,n+1} \quad \text{for any } \mu \in \mathbb{N} \text{ and } v = n, \dots, 2n.$$

Hence we get

$$\begin{aligned}
 mnc_{mn} &\leq \mathcal{C} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) \\
 &+ \sum_{v=n+1}^{2n} \left(\sum_{\mu=m+1}^{2m} \frac{\mathcal{C}}{m} \left(\max_{b_1(\mu) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} c_{jv} \right) + \sum_{\mu=m+1}^{2m} (c_{\mu v} + c_{\mu+1,v}) \right) \\
 &+ \sum_{v=n+1}^{2n} \left(\sum_{\mu=m+1}^{2m} \frac{\mathcal{C}}{m} \left(\max_{b_1(\mu) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} c_{j,v+1} \right) + \sum_{\mu=m+1}^{2m} (c_{\mu,v+1} + c_{\mu+1,v+1}) \right) \\
 &+ \sum_{\mu=m+1}^{2m} \left(\sum_{v=n+1}^{2n} \frac{\mathcal{C}}{n} \left(\max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} c_{\mu k} \right) + \sum_{v=n+1}^{2n} (c_{\mu v} + c_{\mu,v+1}) \right) \\
 &+ \sum_{\mu=m+1}^{2m} \left(\sum_{v=n+1}^{2n} \frac{\mathcal{C}}{n} \left(\max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} c_{\mu+1,k} \right) + \sum_{v=n+1}^{2n} (c_{\mu+1,v} + c_{\mu+1,v+1}) \right) \\
 &\leq \mathcal{C} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) + \mathcal{C} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n} c_{jk} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{jk} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{j+1,k} \\
 &+ \mathcal{C} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n} c_{j,k+1} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{j,k+1} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{j+1,k+1} + \\
 &+ \mathcal{C} \sum_{j=m}^{2m} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{jk} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{j,k+1} \\
 &+ \mathcal{C} \sum_{j=m}^{2m} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{j+1,k} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{j+1,k} + \sum_{j=m}^{2m} \sum_{k=n}^{2n} c_{j+1,k+1} \\
 &\leq \mathcal{C} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) + 2\mathcal{C} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n+1} c_{jk} + 2\mathcal{C} \sum_{j=m}^{2m+1} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 8 \sum_{j=m}^{2m+1} \sum_{k=n}^{2n+1} c_{jk}.
 \end{aligned}$$

This ends the proof. \square

Denote, for $r \in \mathbb{N}$ and $k = 0, 1, 2, \dots$, by

$$\tilde{D}_{k,r}(x) = \frac{\cos(k + \frac{r}{2})x}{2 \sin(\frac{r}{2}x)}$$

the conjugated Dirichlet type kernel.

Lemma 4. ([6], [7]) Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$ and $\{a_k\}_{k=1}^\infty \subset \mathbb{C}$. If $x \neq \frac{2l\pi}{r}$, then for all $m \geq n$

$$\sum_{k=n}^m a_k \sin kx = - \sum_{k=n}^m \Delta_r a_k \tilde{D}_{k,r}(x) + \sum_{k=m+1}^{m+r} a_k \tilde{D}_{k,-r}(x) - \sum_{k=n}^{n+r-1} a_k \tilde{D}_{k,-r}(x)$$

Lemma 5. Let $\{c_{jk}\}_{j,k=1}^\infty \subset \mathbb{C}$ and $m, M, n, N \in \mathbb{N}$ such that $m \leq M$ and $n \leq N$.

(i) If $x \in (0, \frac{\pi}{2})$, then

$$\left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq \frac{\pi}{4x} \left(\sum_{j=m}^M |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \quad (3.10)$$

and if $x \in (\frac{\pi}{2}, \pi)$, then

$$\left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq \frac{\pi}{4(\pi-x)} \left(\sum_{j=m}^M |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right)$$

for any $k \in \mathbb{N}$.

(ii) If $y \in (0, \frac{\pi}{2})$, then

$$\left| \sum_{k=n}^N c_{jk} \sin ky \right| \leq \frac{\pi}{4y} \left(\sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right) \quad (3.11)$$

and if $y \in (\frac{\pi}{2}, \pi)$, then

$$\left| \sum_{k=n}^N c_{jk} \sin ky \right| \leq \frac{\pi}{4(\pi-y)} \left(\sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right)$$

for any $j \in \mathbb{N}$.

Proof. **Part (i).** By Lemma 4, we have

$$\begin{aligned} \left| \sum_{j=m}^M c_{jk} \sin jx \right| &= \left| - \sum_{j=m}^M \Delta_{20} c_{jk} \tilde{D}_{j,2}(x) + \sum_{j=M+1}^{M+2} c_{jk} \tilde{D}_{j,-2}(x) - \sum_{j=m}^{m+1} c_{jk} \tilde{D}_{j,-2}(x) \right| \\ &\leq \sum_{j=m}^M |\Delta_{20} c_{jk}| \cdot |\tilde{D}_{j,2}(x)| + \sum_{j=M+1}^{M+2} |c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| + \sum_{j=m}^{m+1} |c_{jk}| \cdot |\tilde{D}_{j,-2}(x)|. \end{aligned}$$

If $x \in (0, \frac{\pi}{2})$, then using inequality $\sin x \geq \frac{2}{\pi}x$ we obtain the following estimation:

$$\left| \tilde{D}_{j,\pm 2}(x) \right| \leq \left| \frac{\cos(j \pm 1)x}{2 \sin(\pm x)} \right| \leq \frac{1}{2 \sin x} \leq \frac{1}{\frac{2}{\pi}x} \leq \frac{\pi}{4x}. \quad (3.12)$$

From this we get

$$\left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq \frac{\pi}{4x} \left(\sum_{j=m}^M |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right).$$

If $x \in (\frac{\pi}{2}, \pi)$, then using inequality $\sin x \geq 2 - \frac{2}{\pi}x$ we have the estimation:

$$\left| \tilde{D}_{j,\pm 2}(x) \right| \leq \frac{1}{2(2 - \frac{2}{\pi}x)} \leq \frac{\pi}{4(\pi-x)} \quad (3.13)$$

and consequently

$$\left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq \frac{\pi}{4(\pi-x)} \left(\sum_{j=m}^M |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right).$$

Part (ii). Analogously as above

$$\begin{aligned}
 \left| \sum_{k=n}^N c_{jk} \sin ky \right| &= \left| - \sum_{k=n}^N \Delta_{02} c_{jk} \tilde{D}_{k,2}(y) + \sum_{k=N+1}^{N+2} c_{jk} \tilde{D}_{k,-2}(y) - \sum_{k=n}^{n+1} c_{jk} \tilde{D}_{k,-2}(y) \right| \\
 &\leq \sum_{k=n}^N |\Delta_{02} c_{jk}| \cdot |\tilde{D}_{k,2}(y)| + \sum_{k=N+1}^{N+2} |c_{jk}| \cdot |\tilde{D}_{k,-2}(y)| + \sum_{k=n}^{n+1} |c_{jk}| \cdot |\tilde{D}_{k,-2}(y)| \\
 &\leq \frac{\pi}{4y} \left(\sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right)
 \end{aligned}$$

for $y \in (0, \frac{\pi}{2})$, and

$$\left| \sum_{k=n}^N c_{jk} \sin ky \right| \leq \frac{\pi}{4(\pi - y)} \left(\sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right)$$

for $y \in (\frac{\pi}{2}, \pi)$.

This ends the proof. \square

4 Main results

We have the following results:

Theorem 7.

- (i) If a double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C}$ belongs to $DGM(3\alpha, 3\beta, 3\gamma, 2)$ and (2.2) holds, then the regular convergence of double sine series (2.1) is uniform in (x, y) .
- (ii) Conversely, if a double sequence $\{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{R}_+$ belongs to $DGM(2\alpha, 2\beta, 2\gamma, 2)$ and double sine series (2.1) is uniformly regularly convergent in (x, y) , then (2.2) is satisfied.

Theorem 8.

- (i) $DGM(3\alpha, 3\beta, 3\gamma, 1) \subset DGM(3\alpha, 3\beta, 3\gamma, 2)$.
- (ii) There exists a double sequence $\{c_{jk}\}_{j,k=1}^{\infty}$, with the property (2.2), which belongs to the class $DGM(3\alpha, 3\beta, 3\gamma, 2)$ but it does not belong to the class $DGM(3\alpha, 3\beta, 3\gamma, 1)$.

Analogously as in Theorem 8, we can show:

Corollary 1.

- (i) $DGM(2\alpha, 2\beta, 2\gamma, 1) \subset DGM(2\alpha, 2\beta, 2\gamma, 2)$.
- (ii) There exists a double sequence $\{c_{jk}\}_{j,k=1}^{\infty}$, with the property (2.2), which belongs to the class $DGM(2\alpha, 2\beta, 2\gamma, 2)$ but it does not belong to the class $DGM(2\alpha, 2\beta, 2\gamma, 1)$.

Now, we formulate some remarks.

Remark 1. From Theorem 7, using Theorem 8 and Corollary 1, we obtain Theorem 5 and Theorem 4.

Remark 2. There exist $(x_0, y_0) \in \mathbb{R}^2$ and a sequence $\{c_{jk}\}_{j,k=1}^{\infty}$ belonging to the class $DGM(3\alpha, 3\beta, 3\gamma, 3)$, with the property (2.2), such that the series (2.1) is divergent in (x_0, y_0) .

Remark 3. Remark 2 shows that the results from Theorem 7 are not true with $r = 3$ instead of $r = 2$.

5 Proof of the main results

In this section we shall prove our main results.

5.1 Proof of Theorem 7

Part (i): Analogously as in ([7], Theorem 2.5) we can show that single series:

$$\sum_{j=1}^{\infty} c_{jn} \sin jx, \quad n = 1, 2, \dots, \quad \sum_{k=1}^{\infty} c_{mk} \sin ky, \quad m = 1, 2, \dots \quad (5.1)$$

are uniformly convergent since $\{c_{jn}\}_{j=1}^{\infty} \in GM(3\alpha, 2)$ for any $n \in \mathbb{N}$ and $\{c_{mk}\}_{k=1}^{\infty} \in GM(3\beta, 2)$ for any $m \in \mathbb{N}$. Let $\epsilon > 0$ be arbitrarily fixed. We shall prove that for any $M \geq m > \eta$, $N \geq n > \eta$ and any $(x, y) \in \mathbb{R}^2$ we have

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| < (1 + 2\pi\mathcal{C} + 2\pi + 1, 5\pi^2\mathcal{C} + \pi^2)\epsilon, \quad (5.2)$$

where $\eta = \eta(\epsilon) > \lambda$ is the natural number which satisfies for any $m, n > \eta$

$$mn|c_{mn}| < \epsilon, \quad mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22}c_{jk}| < 16C\epsilon, \quad m \sum_{j=m}^{\infty} \sup_{k \geq n} k |\Delta_{20}c_{jk}| < 4C\epsilon, \quad n \sum_{k=n}^{\infty} \sup_{j \geq m} j |\Delta_{02}c_{jk}| < 4C\epsilon.$$

The inequality (5.2) is trivial, when $x = 0$ and y is arbitrary or $y = 0$ and x is arbitrary. We have the same situation, if $x = \pi$ and y is arbitrary or $y = \pi$ and x is arbitrary. Suppose $x, y \in (0, \frac{\pi}{2})$, set $\mu := \lceil \frac{1}{x} \rceil$ and $v := \lceil \frac{1}{y} \rceil$, where $\lceil \cdot \rceil$ means the integer part of a real number. We have four cases:

CASE (a): $\eta < m \leq M < \mu$ and $\eta < n \leq N < v$. Using the inequality $\sin x \leq x$, we have

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq xy \sum_{j=m}^M \sum_{k=n}^N jk |c_{jk}| < \frac{1}{\mu v} \sum_{j=m}^{\mu} \sum_{k=n}^v \epsilon \leq \epsilon.$$

CASE (b): $\max\{\eta, \mu\} < m \leq M$ and $\eta < n \leq N < v$. We obtain

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq \sum_{k=n}^N |\sin ky| \cdot \left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq y \sum_{k=n}^N k \left| \sum_{j=m}^M c_{jk} \sin jx \right|.$$

By (3.10)

$$\begin{aligned} \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| &\leq y \sum_{k=n}^N k \frac{\pi}{4x} \left(\sum_{j=m}^M |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \\ &\leq y \sum_{k=n}^N k \frac{\pi\mu}{4} \left(\sum_{j=m}^M |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \\ &\leq \frac{\pi}{4v} \sum_{k=n}^v \left(m \sup_{k \geq n} k \sum_{j=m}^{\infty} |\Delta_{20}c_{jk}| + 4 \sup_{j \geq m} \sup_{k \geq n} jk |c_{jk}| \right) \end{aligned}$$

and using Lemma 2 and (2.2) we get

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| < \frac{\pi}{4} (4C\epsilon + 4\epsilon) = (\pi\mathcal{C} + \pi)\epsilon.$$

CASE (c): $\eta < m \leq M < \mu$ and $\max\{\eta, v\} < n \leq N$. We have

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq \sum_{j=m}^M |\sin jx| \cdot \left| \sum_{k=n}^N c_{jk} \sin ky \right| \leq x \sum_{j=m}^M j \left| \sum_{k=n}^N c_{jk} \sin ky \right|.$$

Using (3.11)

$$\begin{aligned} \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| &\leq x \sum_{j=m}^M j \frac{\pi}{4y} \left(\sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right) \\ &\leq x \sum_{j=m}^M j \frac{\pi v}{4} \left(\sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right) \\ &\leq \frac{\pi}{4\mu} \sum_{j=m}^{\mu} \left(n \sup_{j \geq m} j \sum_{k=n}^{\infty} |\Delta_{02} c_{jk}| + 4 \sup_{j \geq m} \sup_{k \geq n} jk |c_{jk}| \right) \end{aligned}$$

and by Lemma 2 and (2.2)

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| < \frac{\pi}{4} (4\mathcal{C}\epsilon + 4\epsilon) = (\pi\mathcal{C} + \pi)\epsilon.$$

CASE (d): $\max\{\eta, \mu\} < m \leq M$ and $\max\{\eta, v\} < n \leq N$. Using Lemma 4, we get

$$\begin{aligned} \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| &\leq \left| \sum_{j=m}^M \left(- \sum_{k=n}^N \Delta_{02} c_{jk} \tilde{D}_{k,2}(y) + \sum_{k=N+1}^{N+2} c_{jk} \tilde{D}_{k,-2}(y) - \sum_{k=n}^{n+1} c_{jk} \tilde{D}_{k,-2}(y) \right) \sin jx \right| \\ &= \left| - \sum_{k=n}^N \left(\sum_{j=m}^M \Delta_{02} c_{jk} \sin jx \right) \tilde{D}_{k,2}(y) + \sum_{k=N+1}^{N+2} \left(\sum_{j=m}^M c_{jk} \sin jx \right) \tilde{D}_{k,-2}(y) \right. \\ &\quad \left. - \sum_{k=n}^{n+1} \left(\sum_{j=m}^M c_{jk} \sin jx \right) \tilde{D}_{k,-2}(y) \right| \\ &= \left| - \sum_{k=n}^N \left(- \sum_{j=m}^M \Delta_{02} (\Delta_{20} c_{jk}) \tilde{D}_{j,2}(x) + \sum_{j=M+1}^{M+2} \Delta_{02} c_{jk} \tilde{D}_{j,-2} - \sum_{j=m}^{m+1} \Delta_{02} c_{jk} \tilde{D}_{j,-2}(x) \right) \tilde{D}_{k,2}(y) \right. \\ &\quad + \sum_{k=N+1}^{N+2} \left(- \sum_{j=m}^M \Delta_{20} c_{jk} \tilde{D}_{j,2}(x) + \sum_{j=M+1}^{M+2} c_{jk} \tilde{D}_{j,-2} - \sum_{j=m}^{m+1} c_{jk} \tilde{D}_{j,-2}(x) \right) \tilde{D}_{k,-2}(y) \\ &\quad \left. - \sum_{k=n+1}^{n+1} \left(- \sum_{j=m}^M \Delta_{20} c_{jk} \tilde{D}_{j,2}(x) + \sum_{j=M+1}^{M+2} c_{jk} \tilde{D}_{j,-2} - \sum_{j=m}^{m+1} c_{jk} \tilde{D}_{j,-2}(x) \right) \tilde{D}_{k,-2}(y) \right| \\ &\leq \sum_{j=m}^M \sum_{k=n}^N |\Delta_{22} c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |\tilde{D}_{k,2}(y)| + \sum_{j=M+1}^{M+2} \sum_{k=n}^N |\Delta_{02} c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |\tilde{D}_{k,2}(y)| \\ &\quad + \sum_{j=m}^{m+1} \sum_{k=n}^N |\Delta_{02} c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |\tilde{D}_{k,2}(y)| + \sum_{j=m}^M \sum_{k=N+1}^{N+2} |\Delta_{20} c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |\tilde{D}_{k,-2}(y)| \\ &\quad + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |\tilde{D}_{k,-2}(y)| + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |\tilde{D}_{k,-2}(y)| \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=m}^M \sum_{k=n}^{n+1} |\Delta_{20} c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |\tilde{D}_{k,-2}(y)| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{n+1} |c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |\tilde{D}_{k,-2}(y)| \\
& + \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} |c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |\tilde{D}_{k,-2}(y)|.
\end{aligned}$$

Using (3.12), (2.2) Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
& \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq \\
& \leq \frac{\pi^2}{16xy} \left(\sum_{j=m}^M \sum_{k=n}^N |\Delta_{22} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{j=m}^M \sum_{k=N+1}^{N+2} |\Delta_{20} c_{jk}| \right. \\
& \quad \left. + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^M \sum_{k=n}^{n+1} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{n+1} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} |c_{jk}| \right) \\
& \leq \frac{\pi^2}{16} \left(mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22} c_{jk}| + 4m \sup_{k \geq n} k \sum_{j=m}^{\infty} |\Delta_{20} c_{jk}| + 4n \sup_{j \geq m} j \sum_{k=n}^{\infty} |\Delta_{02} c_{jk}| + 16 \sup_{j \geq m} \sup_{k \geq n} jk |c_{jk}| \right) \\
& < \frac{\pi^2}{16} (16\mathcal{C}\epsilon + 4\mathcal{C}\epsilon + 4\mathcal{C}\epsilon + 16\epsilon) = \left(\frac{3}{2}\mathcal{C} + 1 \right) \pi^2 \epsilon.
\end{aligned}$$

Let $x \in (\frac{\pi}{2}, \pi)$ and $y \in (0, \frac{\pi}{2})$, set $\mu := \left\lceil \frac{1}{\pi-x} \right\rceil$ and $v := \left\lceil \frac{1}{y} \right\rceil$. We have four cases:

CASE (a^*): $\eta < m \leq M < \mu$ and $\eta < n \leq N < v$. Using the inequality $\sin x \leq \pi - x$ and $\sin y \leq y$, we get

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq (\pi - x)y \sum_{j=m}^M \sum_{k=n}^N jk |c_{jk}| < \frac{1}{\mu v} \sum_{j=m}^{\mu} \sum_{k=n}^v \epsilon \leq \epsilon.$$

CASE (b^*): $\max\{\eta, \mu\} < m \leq M$ and $\eta < n \leq N < v$. Applying (3.13) we get

$$\begin{aligned}
\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| & \leq \sum_{k=n}^N |\sin ky| \cdot \left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq y \sum_{k=n}^N k \left| \sum_{j=m}^M c_{jk} \sin jx \right| \\
& \leq y \sum_{k=n}^N k \frac{\pi}{4(\pi-x)} \left(\sum_{j=m}^M |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) < (\pi\mathcal{C} + \pi)\epsilon.
\end{aligned}$$

CASE (c^*): $\eta < m \leq M < \mu$ and $\max\{\eta, v\} < n \leq N$. Analogously as in case (c), we have the inequality

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq \sum_{j=m}^M |\sin jx| \cdot \left| \sum_{k=n}^N c_{jk} \sin ky \right| \leq (\pi - x) \sum_{j=m}^M j \left| \sum_{k=n}^N c_{jk} \sin ky \right| < (\pi\mathcal{C} + \pi)\epsilon.$$

CASE (d^*): $\max\{\eta, \mu\} < m \leq M$ and $\max\{\eta, v\} < n \leq N$. Using (3.12) and (3.13) analogously as in case (d), we obtain

$$\begin{aligned} \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| &\leq \\ &\leq \frac{\pi^2}{16(\pi-x)y} \left(\sum_{j=m}^M \sum_{k=n}^N |\Delta_{22}c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^N |\Delta_{02}c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^N |\Delta_{02}c_{jk}| + \sum_{j=m}^M \sum_{k=N+1}^{N+2} |\Delta_{20}c_{jk}| \right. \\ &\quad \left. + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^M \sum_{k=n}^{n+1} |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{n+1} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} |c_{jk}| \right) \\ &< \left(\frac{3}{2}\mathcal{C} + 1 \right) \pi^2 \epsilon. \end{aligned}$$

Let $x \in (0, \frac{\pi}{2})$ and $y \in (\frac{\pi}{2}, \pi)$, set $\mu := \lceil \frac{1}{x} \rceil$ and $v := \lceil \frac{1}{\pi-y} \rceil$. Now, we have also four cases:

CASE (a^{**}): $\eta < m \leq M < \mu$ and $\eta < n \leq N < v$. Using the inequality $\sin x \leq x$ and $\sin y \leq \pi - y$, we get

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq x(\pi-y) \sum_{j=m}^M \sum_{k=n}^N jk |c_{jk}| < \frac{1}{\mu v} \sum_{j=m}^{\mu} \sum_{k=n}^v \epsilon \leq \epsilon.$$

CASE (b^{**}): $\max\{\eta, \mu\} < m \leq M$ and $\eta < n \leq N < v$. We obtain similarly as in case (b)

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq \sum_{k=n}^N |\sin ky| \cdot \left| \sum_{j=m}^M c_{jk} \sin jx \right| \leq (\pi-y) \sum_{k=n}^N k \left| \sum_{j=m}^M c_{jk} \sin jx \right| < (\pi\mathcal{C} + \pi)\epsilon.$$

CASE (c^{**}): $\eta < m \leq M < \mu$ and $\max\{\eta, v\} < n \leq N$. Applying (3.13) we get

$$\begin{aligned} \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| &\leq \sum_{j=m}^M |\sin jx| \cdot \left| \sum_{k=n}^N c_{jk} \sin ky \right| \leq x \sum_{j=m}^M j \left| \sum_{k=n}^N c_{jk} \sin ky \right| \\ &\leq x \sum_{j=m}^M j \frac{\pi}{4(\pi-y)} \left(\sum_{k=n}^N |\Delta_{02}c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right) < (\pi\mathcal{C} + \pi)\epsilon. \end{aligned}$$

CASE (d^{**}): $\max\{\eta, \mu\} < m \leq M$ and $\max\{\eta, v\} < n \leq N$. Using (3.12) and (3.13) and analogously as in case (d), we get

$$\begin{aligned} \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| &\leq \\ &\leq \frac{\pi^2}{16x(\pi-y)} \left(\sum_{j=m}^M \sum_{k=n}^N |\Delta_{22}c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^N |\Delta_{02}c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^N |\Delta_{02}c_{jk}| + \sum_{j=m}^M \sum_{k=N+1}^{N+2} |\Delta_{20}c_{jk}| \right. \\ &\quad \left. + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^M \sum_{k=n}^{n+1} |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{n+1} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} |c_{jk}| \right) \\ &< \left(\frac{3}{2}\mathcal{C} + 1 \right) \pi^2 \epsilon. \end{aligned}$$

Finally, let $x \in (\frac{\pi}{2}, \pi)$ and $y \in (\frac{\pi}{2}, \pi)$, set $\mu := \lceil \frac{1}{\pi-x} \rceil$ and $v := \lceil \frac{1}{\pi-y} \rceil$. Analogously as before we have four cases:

CASE (a^{***}): $\eta < m \leq M < \mu$ and $\eta < n \leq N < v$. Using the inequality $\sin x \leq \pi - x$ and $\sin y \leq \pi - y$, we get

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq (\pi - x)(\pi - y) \sum_{j=m}^M \sum_{k=n}^N jk |c_{jk}| < \frac{1}{\mu v} \sum_{j=m}^{\mu} \sum_{k=n}^v \epsilon \leq \epsilon.$$

CASE (b^{***}): $\max\{\eta, \mu\} < m \leq M$ and $\eta < n \leq N < v$. We have similarly as in case (b^*)

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq (\pi - y) \sum_{k=n}^N k \left| \sum_{j=m}^M c_{jk} \sin jx \right| < (\pi \mathcal{C} + \pi) \epsilon.$$

CASE (c^{***}): $\eta < m \leq M < \mu$ and $\max\{\eta, v\} < n \leq N$. We obtain similarly as in case (c^{**})

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq (\pi - x) \sum_{j=m}^M j \left| \sum_{k=n}^N c_{jk} \sin ky \right| < (\pi \mathcal{C} + \pi) \epsilon.$$

CASE (d^{***}): $\max\{\eta, \mu\} < m \leq M$ and $\max\{\eta, v\} < n \leq N$. Using (3.12) and (3.13), we have

$$\begin{aligned} & \left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| \leq \\ & \leq \frac{\pi^2}{16(\pi - x)(\pi - y)} \left(\sum_{j=m}^M \sum_{k=n}^N |\Delta_{22} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^N |\Delta_{02} c_{jk}| + \sum_{j=m}^M \sum_{k=N+1}^{N+2} |\Delta_{20} c_{jk}| \right. \\ & \quad \left. + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^M \sum_{k=n}^{n+1} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{n+1} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} |c_{jk}| \right) \\ & < \left(\frac{3}{2} \mathcal{C} + 1 \right) \pi^2 \epsilon \end{aligned}$$

If we summarize all partial estimations we get (5.2), this ends the proof of part (i).

Part (ii): Suppose that $\{c_{jk}\}_{j,k=1}^{\infty}$ is non-negative and let $\epsilon > 0$ be arbitrarily fixed. Using the form (2.2) for the uniform regular convergence of (2.1), we find that there exists an integer $m_0 = m_0(\epsilon)$ for which

$$\left| \sum_{j=m}^M \sum_{k=n}^N c_{jk} \sin jx \sin ky \right| < \epsilon \quad (5.3)$$

holds for any $m + n > m_0$ and any (x, y) . Set $x_1(m) = \frac{\pi}{4m}$, $x_2(m) = \frac{\pi}{4\lambda b_1(m)}$, $y_1(n) = \frac{\pi}{4n}$, $y_2(n) = \frac{\pi}{4\lambda b_2(n)}$ we have

$$\begin{aligned} \sin(jx_1(m)) &\geq \sin \frac{\pi}{4} & \text{if } m \leq j \leq 2m+1; & \quad \sin(jx_2(m)) \geq \sin \frac{\pi}{4\lambda} & \text{if } b_1(m) \leq j \leq 2\lambda b_1(m); \\ \sin(ky_1(n)) &\geq \sin \frac{\pi}{4} & \text{if } n \leq k \leq 2n+1; & \quad \sin(ky_2(n)) \geq \sin \frac{\pi}{4\lambda} & \text{if } b_2(n) \leq k \leq 2\lambda b_2(n). \end{aligned}$$

Since $\{b_1(l)\}_{l=1}^{\infty}$, $\{b_2(l)\}_{l=1}^{\infty}$, $\{b_3(l)\}_{l=1}^{\infty}$ tends to infinity, there exists an m_1 such that for any m, n : $m+n > m_1$ implies $m+n > m_0$, $b_1(m) + n > m_0$, $m + b_2(n) > m_0$ and $b_3(m+n) > m_0$. Then by (5.3) and Lemma 3, we have for $m+n > m_1$

$$\begin{aligned} (5\mathcal{C} + 8)\epsilon &> \mathcal{C} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \sin(jx_1(M)) \sin(ky_1(N)) \right) \\ &+ 2\mathcal{C} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n+1} c_{jk} \sin(jx_2(m)) \sin(ky_1(n)) + 2\mathcal{C} \sum_{j=m}^{2m+1} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} \sin(jx_1(m)) \sin(ky_2(n)) \\ &+ 8 \sum_{j=m}^{2m+1} \sum_{k=n}^{2n+1} c_{jk} \sin(jx_1(m)) \sin(ky_1(n)). \end{aligned}$$

Next

$$(5\mathcal{C} + 8)\epsilon > \mathcal{C} \left(\sin \frac{\pi}{4} \right)^2 \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right) + 2\mathcal{C} \sin \frac{\pi}{4\lambda} \sin \frac{\pi}{4} \sum_{j=b_1(m)}^{2\lambda b_1(m)} \sum_{k=n}^{2n+1} c_{jk} \\ + 2\mathcal{C} \sin \frac{\pi}{4} \sin \frac{\pi}{4\lambda} \sum_{j=m}^{2m+1} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 8 \left(\sin \frac{\pi}{4} \right)^2 \sum_{j=m}^{2m+1} \sum_{k=n}^{2n+1} c_{jk}$$

and finally, we have

$$(5\mathcal{C} + 8)\epsilon > \left(\sin \frac{\pi}{4} \sin \frac{\pi}{4\lambda} \right) mnc_{mn} \quad \text{whenever} \quad m+n > m_1 : \text{ and } : m, n > \lambda.$$

Hence (2.2) is satisfied when $j+k \rightarrow \infty$ and $j, k \geq \lambda$. If $j \rightarrow \infty$ and $k < \lambda$ or $j < \lambda$ and $k \rightarrow \infty$, (2.2) follows from the uniform convergence of the series in (5.1). It completes the proof of part (ii). \square

5.2 Proof of Theorem 8

Part (i): We prove that $DGM_{(3\alpha,3\beta,3\gamma,1)} \subset DGM_{(3\alpha,3\beta,3\gamma,2)}$. Let $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM_{(3\alpha,3\beta,3\gamma,1)}$. It easy to see that

$$\sum_{j=m}^{2m-1} |\Delta_{20}c_{jn}| \leq \sum_{j=m}^{\infty} |\Delta_{20}c_{jn}| = \sum_{j=m}^{\infty} |c_{jn} - c_{j+1,n} + c_{j+1,n} - c_{j+2,n}| \\ \leq \sum_{j=m}^{\infty} |c_{jn} - c_{j+1,n}| + \sum_{j=m}^{\infty} |c_{j+1,n} - c_{j+2,n}| \leq 2 \sum_{j=m}^{\infty} |c_{jn} - c_{j+1,n}| = 2 \sum_{j=m}^{\infty} |\Delta_{10}c_{jn}|$$

and by (2.9), for $m \geq \lambda$ and $n \geq 1$,

$$\sum_{j=m}^{\infty} |\Delta_{10}c_{jn}| = \sum_{r=0}^{\infty} \sum_{j=2^r m}^{2^{r+1}m-1} |\Delta_{10}c_{jn}| \leq \sum_{r=0}^{\infty} \frac{\mathcal{C}}{2^r m} \left(\sup_{M \geq b(2^r m)} \sum_{j=M}^{2M} |c_{jn}| \right) \\ \leq \sum_{r=0}^{\infty} \frac{1}{2^r} \left(\frac{\mathcal{C}}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}| \right) = 2 \cdot \frac{\mathcal{C}}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}|.$$

Next, we get

$$\sum_{j=m}^{2m-1} |\Delta_{20}c_{jn}| \leq 4 \cdot \frac{\mathcal{C}}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}|. \quad (5.4)$$

Similarly as above, for $n \geq \lambda$ and $m \geq 1$,

$$\sum_{k=n}^{2n-1} |\Delta_{02}c_{mk}| \leq 4 \cdot \frac{\mathcal{C}}{n} \sup_{N \geq b(n)} \sum_{k=N}^{2N} |c_{mk}|. \quad (5.5)$$

Now, we have

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22}c_{jk}| = \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11}c_{jk} + \Delta_{11}c_{j+1,k} + \Delta_{11}c_{j,k+1} + \Delta_{11}c_{j+1,k+1}| \\ \leq \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}c_{jk}| + \sum_{j=m+1}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}c_{jk}| + \sum_{j=m}^{\infty} \sum_{k=n+1}^{\infty} |\Delta_{11}c_{jk}| + \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} |\Delta_{11}c_{jk}| \\ \leq 4 \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11}c_{jk}|$$

and by (2.11), we obtain for $m, n \geq \lambda$

$$\begin{aligned} \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{11} c_{jk}| &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{j=2^r m}^{2^{r+1}m-1} \sum_{k=2^s n}^{2^{s+1}n-1} |\Delta_{11} c_{jk}| \leq \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{C}{2^r m \cdot 2^s n} \left(\sup_{M+N \geq b(2^r m + 2^s n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right) \\ &\leq 4 \cdot \frac{C}{mn} \left(\sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right). \end{aligned}$$

Finally, we get

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22} c_{jk}| \leq 16 \cdot \frac{C}{mn} \left(\sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right). \quad (5.6)$$

From (5.4), (5.5) and (5.6) we have that $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM(3\alpha, 3\beta, 3\gamma, 2)$.

Part (ii): Let

$$c_{jk} = \frac{2 + (-1)^j}{j^2} \cdot \frac{2 + (-1)^k}{k^2} \quad \text{for } j, k \in \mathbb{N}. \quad (5.7)$$

We show that $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM(3\alpha, 3\beta, 3\gamma, 2)$. It is easy to see that

$$\Delta_{20} c_{jk} = c_{jk} - c_{j+2,k} = c_{jk} \cdot \frac{4(j+1)}{(j+2)^2}$$

and

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{20} c_{jn}| &= \sum_{j=m}^{2m-1} \left| c_{jn} \cdot \frac{4(j+1)}{(j+2)^2} \right| \leq \sum_{j=m}^{2m-1} |c_{jn}| \frac{4}{j+1} \leq 4 \sum_{j=m}^{2m-1} |c_{jn}| \frac{1}{j} \leq \frac{4}{m} \sum_{j=m}^{2m-1} |c_{jn}| \leq \frac{4}{m} \sum_{j=m}^{2m} |c_{jn}| \\ &\leq \frac{4}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}|. \end{aligned}$$

Similarly as above

$$\Delta_{02} c_{jk} = c_{jk} - c_{j,k+2} = c_{jk} \cdot \frac{4(k+1)}{(k+2)^2}$$

and

$$\sum_{k=n}^{2n-1} |\Delta_{02} c_{mk}| \leq \frac{4}{n} \sup_{N \geq b(n)} \sum_{k=N}^{2N} |c_{mk}|.$$

By elementary calculations

$$\Delta_{22} c_{jk} = c_{jk} - c_{j+2,k} - c_{j,k+2} + c_{j+2,k+2} = c_{jk} \cdot \frac{4(j+1)}{(j+2)^2} \cdot \frac{4(k+1)}{(k+2)^2}$$

and

$$\begin{aligned} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22} c_{jk}| &= \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} \left| c_{jk} \cdot \frac{4(j+1)}{(j+2)^2} \cdot \frac{4(k+1)}{(k+2)^2} \right| \leq 16 \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |c_{jk}| \cdot \frac{1}{j+1} \cdot \frac{1}{k+1} \\ &\leq 16 \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |c_{jk}| \frac{1}{jk} \leq \frac{16}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |c_{jk}| \leq \frac{16}{mn} \sum_{j=m}^{2m} \sum_{k=n}^{2n} |c_{jk}| \\ &\leq \frac{16}{mn} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}|. \end{aligned}$$

Therefore $\{c_{jk}\}_{j,k=1}^\infty \in DGM(3\alpha, 3\beta, 3\gamma, 2)$. Now, we show that $\{c_{jk}\}_{j,k=1}^\infty \notin DGM(3\alpha, 3\beta, 3\gamma, 1)$. We have

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| &= \sum_{j=m}^{2m-1} \left| \frac{(-1)^j + 2}{j^2} \cdot \frac{(-1)^n + 2}{n^2} - \frac{(-1)^{j+1} + 2}{(j+1)^2} \cdot \frac{(-1)^n + 2}{n^2} \right| \\ &= \frac{(-1)^n + 2}{n^2} \sum_{j=m}^{2m-1} \left| \frac{(-1)^j + 2}{j^2} - \frac{2 - (-1)^j}{(j+1)^2} \right|. \end{aligned}$$

Let $A_m = \{j : m \leq j \leq 2m-1 \text{ and } j \text{ is even}\}$. Then

$$\sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| \geq \frac{1}{n^2} \sum_{j \in A_m} \left(\frac{3}{j^2} - \frac{1}{(j+1)^2} \right) \geq \frac{1}{n^2} \sum_{j \in A_m} \left(\frac{3}{j^2} - \frac{1}{j^2} \right) \geq \frac{2}{n^2} \sum_{j \in A_m} \frac{1}{j^2} \geq \frac{m-1}{2n^2m^2}$$

and since

$$\begin{aligned} \frac{\mathcal{C}}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}| &= \frac{\mathcal{C}}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \left| \frac{(-1)^j + 2}{j^2} \cdot \frac{(-1)^n + 2}{n^2} \right| \leq \frac{\mathcal{C}}{m} \frac{(-1)^n + 2}{n^2} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \left| \frac{(-1)^j + 2}{j^2} \right| \\ &\leq \frac{3\mathcal{C}}{mn^2} \sup_{M \geq b(m)} \sum_{j=M}^{2M} \frac{3}{j^2} \leq \frac{9\mathcal{C}}{mn^2} \sup_{M \geq b(m)} \frac{2}{M} \leq \frac{18\mathcal{C}}{mn^2b(m)} \end{aligned}$$

the inequality

$$\sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| \leq \frac{\mathcal{C}}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}|$$

does not hold, because $\frac{1}{b(m)} \rightarrow 0$ as $m \rightarrow \infty$.

This ends the proof. \square

5.3 Proof of Corollary 1

Part (i): We prove that $DGM(2\alpha, 2\beta, 2\gamma, 1) \subset DGM(2\alpha, 2\beta, 2\gamma, 2)$. Let $\{c_{jk}\}_{j,k=1}^\infty \in DGM(2\alpha, 2\beta, 2\gamma, 1)$. Then for $m \geq \lambda$ and $n \geq 1$

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{20}c_{jn}| &= \sum_{j=m}^{2m-1} |\Delta_{10}c_{jn} + \Delta_{10}c_{j+1,n}| \leq \sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| + \sum_{j=m}^{2m-1} |\Delta_{10}c_{j+1,n}| \\ &\leq \sum_{j=m}^{2m-1} |\Delta_{10}c_{jn}| + \sum_{j=m+1}^{2m} |\Delta_{10}c_{jn}| \leq 2 \sum_{j=m}^{2m} |\Delta_{10}c_{jn}| + |\Delta_{10}c_{2m,n}| \\ &\leq 2 \frac{\mathcal{C}}{m} \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} c_{jn} + \sum_{k=2m}^{4m-1} |\Delta_{10}c_{jn}| \\ &\leq 2 \frac{\mathcal{C}}{m} \left(\max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} c_{jn} + \max_{b_1(2m) \leq M \leq \lambda b_1(2m)} \sum_{j=M}^{2M} c_{jn} \right) = 2 \frac{\mathcal{C}}{m} \{S_1 + S_2\}. \end{aligned}$$

Let, for $m \geq \lambda$, $b'_1(m) = b_1(m)$ if $S_2 \leq S_1$ or $b'_1(m) = b_1(2m)$ if $S_1 \leq S_2$. Then

$$\sum_{j=m}^{2m-1} |\Delta_{20}c_{jn}| \leq 4 \frac{\mathcal{C}}{m} \max_{b'_1(m) \leq M \leq \lambda b'_1(m)} \sum_{j=M}^{2m} c_{jn} \quad (5.8)$$

for $m \geq \lambda$ and $n \geq 1$. Similarly as above

$$\sum_{k=n}^{2n-1} |\Delta_{02}c_{mk}| \leq 4 \frac{\mathcal{C}}{n} \max_{b'_2(n) \leq N \leq \lambda b'_2(n)} \sum_{k=N}^{2n} c_{mk}, \quad (5.9)$$

for $n \geq \lambda$ and $m \geq 1$. Finally, by (5.6) we get

$$\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22} c_{jk}| \leq 16 \cdot \frac{C}{mn} \left(\sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right) \quad (5.10)$$

for $m, n \geq \lambda$. From (5.8), (5.9) and (5.10) we get that $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM(2\alpha, 2\beta, 2\gamma, 2)$.

Part (ii): Taking the sequence (5.7), we can show, similarly as in the proof of Theorem 8 (part (ii)) that $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM(2\alpha, 2\beta, 2\gamma, 2)$ and $\{c_{jk}\}_{j,k=1}^{\infty} \notin DGM(2\alpha, 2\beta, 1\gamma, 1)$.

It completes the proof. \square

5.4 Proof of Remark 2

Let $c_{jk} = a_j \cdot a_k$, where

$$a_n = \begin{cases} \frac{3}{n \ln(n+1)} & \text{if } n = 3l + 1, \\ \frac{1}{n \ln(n+1)} & \text{if } n \neq 3l + 1, \end{cases}$$

for $l \in \mathbb{N} \cup \{0\}$ and $(x_0, y_0) = (\frac{2}{3}\pi, \frac{2}{3}\pi)$.

It is easy to see that (2.2) holds. Now, we shall prove that $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM(3\alpha, 3\beta, 3\gamma, 3)$. Let

$$A_m = \{j : j = 3l + 1, l \in \mathbb{N} \cup \{0\}, m \leq j \leq 2m - 1\}$$

and

$$B_m = \{j : j \neq 3l + 1, l \in \mathbb{N} \cup \{0\}, m \leq j \leq 2m - 1\}.$$

Then we have

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{30} c_{jn}| &= \sum_{j=m}^{2m-1} a_n |a_j - a_{j+3}| \\ &= \sum_{j \in A_m} a_n \left(\frac{3}{j \ln(j+1)} - \frac{3}{(j+3) \ln(j+4)} \right) + \sum_{j \in B_m} a_n \left(\frac{1}{j \ln(j+1)} - \frac{1}{(j+3) \ln(j+4)} \right) \\ &= 3 \sum_{j \in A_m} a_n \left(\frac{j(\ln(j+4) - \ln(j+1)) + 3 \ln(j+4)}{j(j+3) \ln(j+1) \ln(j+4)} \right) \\ &\quad + \sum_{j \in B_m} a_n \left(\frac{j(\ln(j+4) - \ln(j+1)) + 3 \ln(j+4)}{j(j+3) \ln(j+1) \ln(j+4)} \right). \end{aligned}$$

Applying the Lagrange theorem, for the function $y = \ln(x)$, there exists $c \in (j+4, j+1)$ such that

$$\ln(j+4) - \ln(j+1) = \frac{3}{c} \leq \frac{3}{j+1}$$

and

$$\begin{aligned} \sum_{j=m}^{2m-1} |\Delta_{30} c_{jn}| &\leq 3 \sum_{j \in A_m} a_n \left(\frac{\frac{3j}{j+1} + 3 \ln(j+4)}{j(j+3) \ln(j+1) \ln(j+4)} \right) + \sum_{j \in B_m} a_n \left(\frac{\frac{3j}{j+1} + 3 \ln(j+4)}{j(j+3) \ln(j+1) \ln(j+4)} \right) \\ &\leq 3 \sum_{j \in A_m} a_n \left(\frac{3 + 3 \ln(j+4)}{j^2 \ln(j+1) \ln(j+4)} \right) + \sum_{j \in B_m} a_n \left(\frac{3 + 3 \ln(j+4)}{j^2 \ln(j+1) \ln(j+4)} \right) \\ &\leq 3 \sum_{j \in A_m} a_n \left(\frac{6}{j^2 \ln(j+1)} \right) + \sum_{j \in B_m} a_n \left(\frac{6}{j^2 \ln(j+1)} \right) = 6 \sum_{j \in A_m} a_n a_j \frac{1}{j} + 6 \sum_{j \in B_m} a_n a_j \frac{1}{j} \\ &\leq \frac{6}{m} \sum_{j=m}^{2m} |c_{jn}| \leq \frac{6}{m} \sup_{M \geq b(m)} \sum_{j=M}^{2M} |c_{jn}| \quad (5.11) \end{aligned}$$

Similarly as above

$$\sum_{k=n}^{2n-1} |\Delta_{03} c_{mk}| \leq \frac{6}{n} \sup_{N \geq b(n)} \sum_{k=N}^{2n} |c_{mk}|. \quad (5.12)$$

By elementary calculations we have

$$\begin{aligned} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{33} c_{jk}| &= \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |(a_j - a_{j+3})(a_k - a_{k+3})| = \sum_{j=m}^{2m-1} |a_j - a_{j+3}| \sum_{k=n}^{2n-1} |a_k - a_{k+3}| \\ &= \left(\sum_{j \in A_m} \left(\frac{3}{j \ln(j+1)} - \frac{3}{(j+3) \ln(j+4)} \right) + \sum_{j \in B_m} \left(\frac{1}{j \ln(j+1)} - \frac{1}{(j+3) \ln(j+4)} \right) \right) \\ &\quad \cdot \left(\sum_{k \in A_n} \left(\frac{3}{k \ln(k+1)} - \frac{3}{(k+3) \ln(k+4)} \right) + \sum_{k \in B_n} \left(\frac{1}{k \ln(k+1)} - \frac{1}{(k+3) \ln(k+4)} \right) \right) \\ &\leq \left(6 \sum_{j \in A_m} \frac{3}{j^2 \ln(j+1)} + 6 \sum_{j \in B_m} \frac{1}{j^2 \ln(j+1)} \right) \left(6 \sum_{k \in A_n} \frac{3}{k^2 \ln(k+1)} + 6 \sum_{k \in B_n} \frac{1}{k^2 \ln(k+1)} \right) \\ &= \left(6 \sum_{j \in A_m} c_j \frac{1}{j} + 6 \sum_{j \in B_m} c_j \frac{1}{j} \right) \left(6 \sum_{k \in A_n} c_k \frac{1}{k} + 6 \sum_{k \in B_n} c_k \frac{1}{k} \right) \leq \frac{36}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |c_{jk}| \\ &\leq \frac{36}{mn} \sup_{M+N \geq b(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}|. \end{aligned} \quad (5.13)$$

From (5.11), (5.12) and (5.13) we obtain that $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM({}_3\alpha, {}_3\beta, {}_3\gamma, 3)$.

Moreover, we have

$$\begin{aligned} &\sum_{j=1}^{3M+2} \sum_{k=1}^{3N+2} c_{jk} \sin jx_0 \sin ky_0 \\ &= \left(a_1 \sin \frac{2}{3}\pi + a_2 \sin \frac{4}{3}\pi + \sum_{j=3}^{3M+2} a_j \sin jx_0 \right) \left(a_1 \sin \frac{2}{3}\pi + a_2 \sin \frac{4}{3}\pi + \sum_{k=3}^{3N+2} a_k \sin ky_0 \right) \\ &= \left[\left(a_1 \sin \frac{2}{3}\pi - a_2 \sin \frac{2}{3}\pi \right) + \sum_{j=1}^M \sum_{i=0}^2 a_{3j+i} \sin \left((3j+i) \frac{2}{3}\pi \right) \right] \\ &\quad \cdot \left[\left(a_1 \sin \frac{2}{3}\pi - a_2 \sin \frac{2}{3}\pi \right) + \sum_{k=1}^N \sum_{i=0}^2 a_{3k+i} \sin \left((3k+i) \frac{2}{3}\pi \right) \right] \\ &= \left[\sin \frac{2}{3}\pi (a_1 - a_2) + \sum_{j=1}^M \left(a_{3j+1} \sin \frac{2}{3}\pi + a_{3j+2} \sin \frac{4}{3}\pi \right) \right] \cdot \left[\sin \frac{2}{3}\pi (a_1 - a_2) + \sum_{k=1}^N \left(a_{3k+1} \sin \frac{2}{3}\pi + a_{3k+2} \sin \frac{4}{3}\pi \right) \right] \\ &= \left(\sin \frac{2}{3}\pi \right)^2 \sum_{j=0}^M \sum_{k=0}^N \left(\frac{3}{(3j+1) \ln(3j+2)} - \frac{1}{(3j+2) \ln(3j+3)} \right) \cdot \left(\frac{3}{(3k+1) \ln(3k+2)} - \frac{1}{(3k+2) \ln(3k+3)} \right) \\ &\geq \left(\sin \frac{2}{3}\pi \right)^2 \sum_{j=0}^M \sum_{k=0}^N \frac{2}{(3j+1) \ln(3j+3)} \cdot \frac{2}{(3k+1) \ln(3k+3)} \rightarrow \infty \quad \text{as} \quad M+N \rightarrow \infty. \end{aligned}$$

This ends the proof. \square

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